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Hamiltonian studies of the two-dimensional n-component cubic model: II. The cubic transition

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Abstract. The phase transition behaviour of the Hamiltonian version of the two-dimensional n-component cubic model is studied along the cubic transition line. $1/n$ expansion and anisotropy expansion around the $2n$-state Potts point are used to determine the phase transition line and the latent heat. The latent heat depends on the value of the coupling whereas the crossover value of n, where the transition changes from second to first order, does not (it is $n_c = 2$). The latent heat has an essential singularity at $n = 2$ along the cubic transition line.

1. Introduction

In the preceding paper (Iglói 1986, hereafter referred to as paper I) the phase diagram and the critical properties of the $(1+1)$-dimensional n-component cubic model were determined. In this paper the properties of the cubic transition are investigated, when this transition is of first order. We use two methods of expansion which supply complementary information. The $1/n$ expansion for large values of n is a good approximation, while the coefficients of the anisotropy expansion around the $2n$-state Potts point have small values when n is close to 2. These expansion methods are unfortunately not suitable for investigating the neighbourhood of the multicritical point because at this point all terms of the perturbational expressions become equally important, signalling the role of competing effects.

The $1/n$ series is determined up to second order for the phase transition line, for the latent heat and for the crossover value of n_c, where the order of the phase transition changes from first to second order. In this paper the $2n$-state Potts model was taken as a reference system, and the expansion was made with respect to the anisotropy. This series was determined in first order only because calculation of the higher-order terms would need a knowledge of the properties of the $2n$-state Potts model beyond the critical point, and these are not known.

The paper is arranged as follows: § 2 contains the formalism and the duality properties of the model; §§ 3 and 4 respectively give the results of the $1/n$ expansion and anisotropy expansion; § 5 contains a summary. The details of the calculations are presented in the appendix.

2. Duality properties

To investigate the duality properties of the model we use the strong-coupling representa-

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tion (paper I, equations (2.1) and (2.2)), and rewrite the Hamiltonian in the form

$$H=-\frac{\lambda_1}{2n}\sum_i\sum_{k=1}^{n}\Omega_i^{2k-1}\Omega_{i+1}^{2n-2k+1}-\frac{\lambda_2}{2n}\sum_i\sum_{k=1}^{n-1}\Omega_i^{2k}\Omega_{i+1}^{2n-2k}$$

$$-h_1\sum_i\sum_{k=1}^{n-1}(M_i^k+M_i^{n+k})-h_2\sum_i M_i^n. \tag{2.1}$$

Here λ_1 and λ_2 denote the couplings between the neighbouring spins, while h_1 and h_2 are the strengths of the external fields appearing in the Hamiltonian version of the model. Ω and M are $2n\times 2n$ matrices:

$$\Omega=\begin{bmatrix}1 & 0 & & & 0\\ 0 & \omega & & & \\ & & \omega^2 & & \\ & & & \ddots & \\ & & & & 0\\ 0 & \dots & & 0 & \omega^{2n-1}\end{bmatrix},\qquad M=\begin{bmatrix}0 & 0 & \dots & & 0 & 1\\ 1 & 0 & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & \ddots & & \\ 0 & & & & 1 & 0\end{bmatrix},$$

and $\omega=\exp(2\pi\mathrm{i}/2n)$. These matrices commute in different sites; on the same site they satisfy the $Z(2n)$algebra:

$$\Omega_i^k\Omega_i^l=\Omega_i^{k+l},\qquad M_i^kM_i^l=M_i^{k+l},\qquad M_i^k\Omega_i^l=\omega^{kl}\Omega_i^lM_i^k. \tag{2.2}$$

Let us now introduce a dual lattice and define the operators Ω_i^k, M_i^k on the sites of the dual lattice, i.e. on the links of the original lattice:

$$\bar\Omega_i^k=\prod_{j\le i}M_j^k,\qquad \bar M_i^k=\Omega_i^{2n-k}\Omega_{i+1}^k.$$

It is easy to see that these operators also satisfy the $Z(2n)$ algebra. The Hamiltonian operator (2.1) can be written in terms of the new operators as

$$H=-\frac{\lambda_1}{2n}\sum_i\sum_{k=1}^{n}\bar M_i^{2k-1}-\frac{\lambda_2}{2n}\sum_i\sum_{k=1}^{n-1}\bar M_i^{2k}$$

$$-h_1\sum_i\sum_{k=1}^{n-1}(\bar\Omega_i^k\bar\Omega_{i+1}^{2n-k}+\bar\Omega_i^{n+k}\bar\Omega_i^{n-k})-h_2\sum_i\bar\Omega_i^n\bar\Omega_{i+1}^n. \tag{2.3}$$

Comparing (2.1) and (2.3), one can see that the model is not, in general, self-dual. However, by using the parametrisation

$$\frac{\lambda_1}{2n}=1,\qquad h=\frac{2nh_1}{\lambda_1}=\frac{2nh_2}{\lambda_2},\qquad \lambda=\frac{\lambda_2}{\lambda_1}=\frac{h_2}{h_1}, \tag{2.4}$$

the Hamiltonian can be written as

$$H=H_0+H_{\mathrm{p}},$$

$$H_0=-\sum_i\sum_{k=1}^{2n-1}\Omega_i^k\Omega_{i+1}^k-h\sum_i\sum_{k=1}^{2n-1}M_i^k-(\lambda-1)\left(\sum_i\sum_{k=1}^{n-1}\Omega_i^{2k}\Omega_{i+1}^{2n-2k}+h\sum_i\sum_{k=1}^{n-1}M_i^{2k}\right), \tag{2.5}$$

$$H_{\mathrm{p}}=(\lambda-1)h\sum_i\left(\sum_{k=1}^{n-1}M_i^{2k}-M_i^n\right).$$

H_0 obeys the duality relation

$$H_0(h,\lambda)=hH_0(h^{-1},\lambda)$$

and its self-dual line is $h = 1$ independently of λ. The original model (described by H) is self-dual if $H_p = 0$. This is true for the Ashkin-Teller model, where $n = 2$, and at the $2n$-state Potts point, i.e. at $\lambda = 1$. The plane given by equation (2.4) was used by Kohmoto *et al* (1981) in their study on the Ashkin-Teller model. It turned out that this subspace contains all the interesting regions of the phase diagram (Iglói and Sólyom 1984). Finally we would mention that the (2.5) form of the Hamiltonian will be used in § 4 to perform an anisotropy expansion around the $2n$-state Potts point in powers of $\lambda - 1$.

3. $1/n$ expansion

For several spin systems, in which the number of components of the spin is a parameter and the $n \to \infty$ limit is exactly soluble, a well converging $1/n$ expansion can be defined. The method was introduced by Kogut (1980) and was applied to several spin and gauge systems. The success of the method is due to the fact that even in first order, infinite terms of different orders of the usual Brillouin-Wigner perturbation series have to be summed.

The essence of the $1/n$ expansion may be summarised as follows. The ground state energy of the system as a function of the coupling has to be determined for different powers of $1/n$, both in the weak-coupling and the strong-coupling regime. The crossing point for the two expressions is identified with the phase transition point in the given order. Furthermore, the difference in the slopes of the two curves at the crossing point is proportional to the latent heat, also in the given order of $1/n$. The latent heat defined in this way is positive for large values of n; however, it becomes negative with decreasing values. As was pointed out by Kogut (1980), the polynomial form of the finite series is not able to account for the essential singularity in the latent heat corresponding to the second- to first-order change in the transition. Therefore the latent heat obtained by $1/n$ expansion has a physical meaning only for its positive values.

The crossover value of n is defined by the zero of the latent heat expression, while for its negative values the transition is assumed to be of second order. The estimate for the crossover value of n turned out to be fairly accurate even in first order for the Potts model in two and three dimensions (Kogut *et al* 1980, Kogut and Sinclair 1981). The further terms give a slight improvement though the convergence of the series is rather slow. For illustration, the approximate crossover values of n are summarised in table 1 for the n-state Potts model in two dimensions. Here, the exact solution of Baxter was used both for the latent heat (Baxter 1973) and for the jump in the magnetisation (Baxter 1982):

$$L = 1 - \frac{6}{n} + \frac{2}{n^2} + \frac{8}{n^3} + \frac{26}{n^4} + \frac{76}{n^5} + \ldots,$$

$$\Delta M = 1 - \frac{1}{n} - \frac{3}{n^2} - \frac{9}{n^3} - \frac{27}{n^4} - \frac{82}{n^5} \ldots.$$

From table 1 it can be seen that the two series for n_c go from different directions to the exact value of 4.

The $1/n$ expansion is easier to apply for systems where self-duality holds. In these cases the weak-coupling and strong-coupling series are connected by self-duality, and the approximate transition points coincide with the self-dual point in every order of the calculation.

Table 1. Zeros of the latent heat and the magnetisation series in powers of $1/n$ for the 2D n-state Potts model.

	Crossover value of n	
Order	Latent heat	Magnetisation
1	6	1
2	5.646	2.303
3	5.346	3
4	5.106	3.378
5	4.911	3.598

In our model, where self-duality generally does not hold, the two expansions have to be carried out separately. The first few terms of the weak- and strong-coupling series are given in the appendix. In the neighbourhood of the phase transition points the terms of the series can be arranged in powers of $1/n$ if the condition

$$2nh_1^2 \gg h_2^2 \tag{3.1}$$

is fulfilled. This condition is true for the cubic transition line. Unfortunately, the multicritical point cannot be investigated by this method. In that region all terms of the series become of the same order of magnitude, a fact that clearly represents the role of competition in creating multicritical points. In the following the phase diagram and the latent heat of the cubic transition are given up to second order.

The ground state energy in the strong- and weak-coupling limit up to second order in $1/n$ is given in the appendix. The phase transition point is given as

$$2h = 1+\lambda+\frac{1}{2n}\alpha_1(\lambda)+\frac{1}{(2n)^2}\alpha_2(\lambda)+\ldots. \tag{3.2}$$

Here

$$\alpha_1(\lambda) = -(\lambda-1)[2+\lambda+(\lambda-1)/(\lambda+3)]$$

while the form of $\alpha_2(\lambda)$ is given in figure 1. The latent heat is

$$L = 2nh\left(1+\frac{1}{2n}\delta_1(\lambda)+\frac{1}{(2n)^2}\delta_2(\lambda)+\ldots\right) \tag{3.3}$$

where

$$\delta_1(\lambda) = -6+(\lambda-1)-2[(\lambda-1)/(\lambda+3)]^2$$

while $\delta_2(\lambda)$ is sketched in figure 1.

We also give the results in the original space of couplings up to first order, when the formulae are relatively simple:

$$4nh_1 = \lambda_1+\lambda_2+\frac{1}{2n}\left(-4nh_2+2\lambda_1-\frac{(\lambda_2-\lambda_1)^2}{\lambda_2+3\lambda_1}\right)+\sigma\left(\frac{1}{n^2}\right), \tag{3.4}$$

$$L = 2nh_1\left\{1-\frac{1}{2n}\left[7-\frac{h_2}{h_1}+2\left(\frac{\lambda_2-\lambda_1}{\lambda_2+3\lambda_1}\right)^2\right]+\sigma\left(\frac{1}{n^2}\right)\right\}. \tag{3.5}$$

In the following let us discuss some consequences of these formulae.

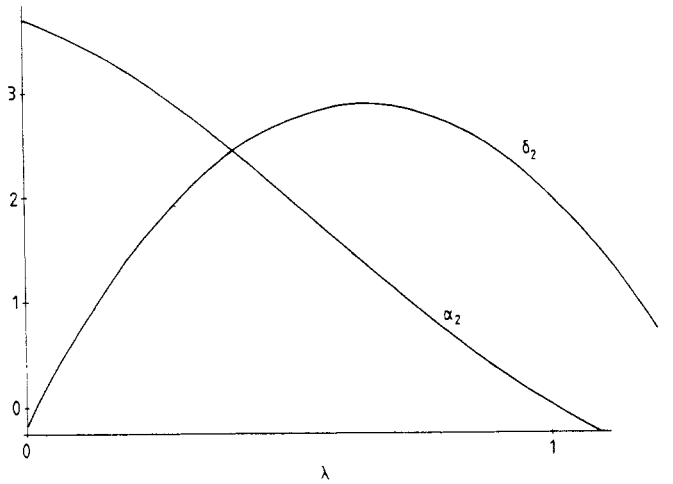

Figure 1. Second-order expansion parameters of the phase transition point (α_2) and of the latent heat (δ_2).

(i) In zeroth order, i.e. in the $n \to \infty$ limit, the phase transition points (3.2) and (3.4) are the same as in the mean-field calculation (§ 3, paper I). Therefore the mean-field phase diagram is exact for the cubic transition line in the $n \to \infty$ limit.

(ii) The position of the multicritical point for large values of n can be obtained from condition (3.1). It is

$$2nh_1^2 \approx h_2^2.$$

This expression is also in agreement with the results of mean-field calculation.

(iii) The latent heat depends on the values of the couplings, even in first order of $1/n$. Therefore there is no universality for the first-order transition.

(iv) In order to investigate the convergence of the series (3.2) for the phase transition points, we compare it with the results of other methods. At $\lambda = 1$ (at the $2n$-state Potts point) equation (3.2) is exact. In another part of the phase diagram, at $\lambda = 0$, we compare it with the result of the self-dual RG calculation (§ 4, paper I). Figure 2 shows that the results of the two methods are quite close to each other for $n \gtrsim 3$.

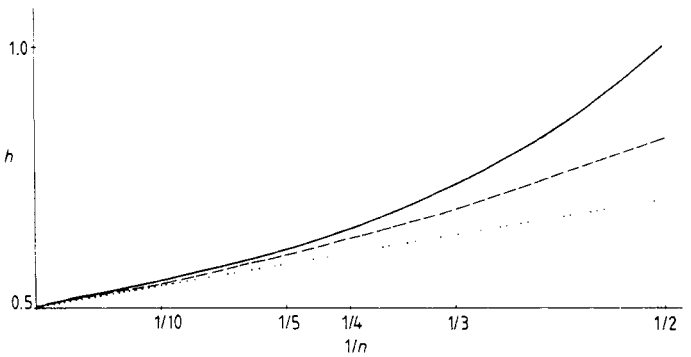

Figure 2. Phase transition point at $\lambda = 0$, for different values of n. ——, RG calculation; - - -, $1/n$ expansion in second order; · · ·, $1/n$ expansion in first order.

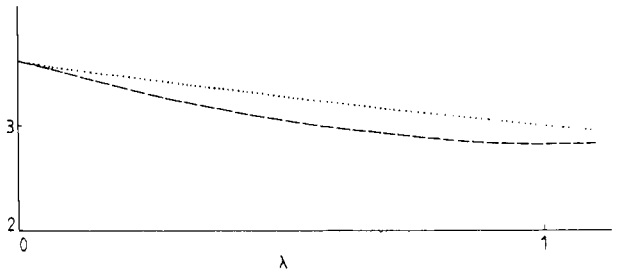

Figure 3. The critical values of n_c where the nature of the transition changes. - - -, second-order calculation; · · · ·, first-order calculation.

(v) As already mentioned, the latent heat expression gives the possibility to estimate the crossover values of n_c, where the phase transition changes from first order to second order. The estimated n_c values for different values of λ are given in figure 3. It is generally supposed (Nienhuis *et al* 1983) that the exact value of n_c is 2 independently of the value of λ. In our case, even in this low order of the calculation, the estimated values of n_c are in the range 2.8–3.6, and there is no strong dependence of n_c on λ. So this picture can be considered to be consistent with the conjectured one.

(vi) By expanding equations (3.2) and (3.3) around the $2n$-state Potts point in powers of $(\lambda-1)$, the following series can be obtained:

$$2h = 2+(\lambda-1)\left(1-\frac{3}{2n}-\frac{3}{(2n)^2}\ldots\right)+(\lambda-1)^2\left(-\frac{5}{4}\frac{1}{2n}+\frac{27}{8}\frac{1}{(2n)^2}\ldots\right)+\ldots, \tag{3.6}$$

$$L = 2nh\left[1-\frac{6}{2n}+\frac{2}{(2n)^2}+\ldots+(\lambda-1)\left(\frac{1}{2n}-\frac{5}{(2n)^2}\ldots\right)+\ldots\right]. \tag{3.7}$$

It will be shown in § 4 that in first order of $(\lambda-1)$ the coefficients can be summed. This exercise will be done by using anisotropy expansion around the $2n$-state Potts point.

4. $(\lambda-1)$ expansion around the $2n$-state Potts point

Let us turn back to expression (2.5), where the Hamiltonian of the system is split into a self-dual part (H_0) and a perturbation (H_p). Let us suppose that $n>2$. The phase transition is of first order so the transition is accompanied by a crossing of energy levels. Let us denote the difference between the two lowest levels by $F(h,\lambda-1)$, and the same quantities for H_0 and H_p are denoted by F_0 and F_p, respectively. The gap vanishes at the phase transition point, so

$$F(h^*,\lambda-1)=0. \tag{4.1}$$

In the $2n$-state Potts point, and for the system described by the Hamiltonian H_0, the transition point is just $h^*=1$, therefore

$$F(1,0)=0, \qquad F_0(1,\lambda-1)=0.$$

Now let us suppose that it is possible to expand $F(h^*,\lambda-1)$ in equation (4.1) with respect to $(\lambda-1)$. This assumption is non-trivial and we cannot prove it; however, the final results are taken as retrospective evidence for the validity of the method.

By expanding (4.1) in first order in $(\lambda-1)$,

$$F(h^*, \lambda-1) = F_0(1, \lambda-1) + \left.\frac{\partial F_0}{\partial h}\right|_{\substack{\lambda=1 \\ h=1}} (h^*-1) + F_p(1, \lambda-1) + \ldots = 0.$$

Thus the change in the phase transition point is

$$h^*-1 = -\frac{F_p(1,0)}{(\partial/\partial h)F_0(h,0)}. \tag{4.2}$$

The denominator on the right-hand side of (4.2) is proportional to the latent heat of the $2n$-state Potts model, that can be written by using the Hellmann-Feynman theorem as

$$-\frac{\partial}{\partial h} F_0(h,0) = \left\langle 1 \left| \sum_i \sum_{k=1}^{2n-1} M_i^k \right| 1 \right\rangle - \left\langle 0 \left| \sum_i \sum_{k=1}^{2n-1} M_i^k \right| 0 \right\rangle \tag{4.3}$$

where $|0\rangle$ and $|1\rangle$ denote the ground state and the first excited states of the $2n$-state Potts model, respectively. The numerator of (4.2) has a form similar to (4.3), namely

$$F_p(1,0) = (\lambda-1)\left[\left\langle 1 \left| \sum_i \left(\sum_{k=1}^{n-1} M_i^{2k} - M_i^n \right) \right| 1 \right\rangle - \left\langle 0 \left| \sum_i \left(\sum_{k=1}^{n-1} M_i^{2k} - M_i^n \right) \right| 0 \right\rangle \right].$$

Since both in the disordered phase and in the broken symmetry phase the symmetry of the Potts model requires that $\langle M^k \rangle$ is independent of (non-zero) k, the two expressions are proportional to each other and their ratio in (4.2) is just $(n-2)/(2n-1)\ (\lambda-1)$.

Thus, the phase transition points in first order are given by

$$h^* = 1 + \frac{n-2}{2n-1}(\lambda-1) + \ldots. \tag{4.4}$$

This result coincides with the result of the $1/n$ expansion in equation (3.6). It is pointed out that equation (4.4) holds for the single phase transition part of the phase diagram, otherwise it may be considered as a 'duality line'.

The latent heat can be calculated from the slope of the $F(h, \lambda-1)$ curve at h^*:

$$L = -\partial F(h, \lambda)/\partial h|_{h^*}.$$

By expanding L in first order of $\lambda-1$,

$$L = -\left.\frac{\partial F_0(h,\lambda)}{\partial h}\right|_{h=h^*} - \left.\frac{\partial F_p}{\partial h}\right|_{h=1}. \tag{4.5}$$

The first term can be expanded as

$$-\left.\frac{\partial F_0(h,\lambda)}{\partial h}\right|_{h=h^*} = -\left.\frac{\partial F_0(h,1)}{\partial h}\right|_{h=1} - \left.\frac{\partial^2 F_0(h,1)}{\partial h^2}\right|_{h=1} (h^*-1) - \left.\frac{\partial^2 F_0(h,\lambda)}{\partial h \partial \lambda}\right|_{\substack{h=1 \\ \lambda=1}} (\lambda-1) + \ldots. \tag{4.6}$$

The first term is just the latent heat of the $2n$-state Potts model; the second is zero due to the duality properties of the Potts model. The third term of (4.6) as well as the

second term in (4.5) are proportional to the latent heat of the $2n$-state Potts model. Therefore we can write up to first order

$$L = L_{\text{Potts}}[1+(2n-1)^{-1}(\lambda-1)+\ldots]. \tag{4.7}$$

This formula is also in accordance with the result of the $1/n$ expansion (3.7).

Now supposing that this series does not contain a diverging coefficient in some finite order, and the series has finite sum, then equation (4.7) gives compelling evidence that the cubic phase transition turns first order at $n=2$ for a non-zero range of λ, and the latent heat has an essential singularity at $n=2$. The non-fulfilment of these conditions, however, would not be compelling counterevidence, since the latent heat of the cubic transition might vanish at $n=2$ at an infinitely faster or slower rate than that of the Potts model. Finally it is mentioned that the higher-order terms of the anisotropy expansion should be exposed by the derivatives of the free energy of the Potts model at the critical point; however, these are not yet known.

5. Summary

In this paper, which represents the second part of our work on the (1+1)D n-component cubic model, the properties of the system were investigated along the cubic transition line, for the first-order transition. We used the $1/n$ expansion and anisotropy expansion, which supply complementary information. The results show the cubic transition to be of first order for $n>2$, independently of the coupling. The latent heat depends on the coupling (there is no universality), and has an essential singularity at $n=2$. At the multicritical point all terms of the series are of the same order of magnitude, which prevents the application of a perturbational expansion.

Acknowledgment

The author is indebted to J Sólyom for useful discussions and grateful to an unknown referee for useful recommendations.

Appendix

Strong-coupling expansion

The strong-coupling representation (paper I, equations (2.1) and (2.2)) is used in the calculation. The unperturbed part of the Hamiltonian is

$$H_0 = H_\lambda + \sum_i h_2$$

while the perturbation

$$V = H_h - \sum_i h_2.$$

The ground state of H_0 is given by

$$\psi_0^{s,0} = |1\,1\ldots1\rangle$$

with the energy $E_0^{s,0} = -N\lambda_1/2 + Nh_2$.

The lower lying excitations of H_0 contain one flipped spin, and they are of two kinds:

$$\psi^{s,0}_{1,k} = |1\ 1 \ldots 1\ k\ 1 \ldots 1\rangle$$

with energy $E^{s,0}_{1,k} = E^{s,0}_0 + \lambda_1 + \lambda_2$, and

$$\psi^{s,0}_{1,n+1} = |1\ 1 \ldots 1\ (n+1) 1 \ldots 1\rangle$$

with energy $E^{s,0}_{1,n+1} = E^{s,0}_0 + 2\lambda_1$. The two excitations are $N(2n-2)$- and N-fold degenerate, respectively. The higher lying excitations of H_0 contain more flipped spins.

The first few terms of the Brillouin-Wigner perturbational series for the ground state energy are the following:

$$E^{s,0}_0 = -N(\lambda_1/2 + h_2), \qquad E^{s,1}_0 = 0,$$

$$E^{s,2}_0 = -N\left((2n-2)\frac{h_1^2}{\lambda_1+\lambda_2} + \frac{h_2^2}{2\lambda_1}\right),$$

$$E^{s,3}_0 = -N\left((2n-2)(2n-4)\frac{h_1^3}{(\lambda_1+\lambda_2)^2} + 2(2n-2)\frac{h_1^2 h_2}{(\lambda_1+\lambda_2)2\lambda_1} + (2n-2)\frac{h_1^2 h_2}{(\lambda_1+\lambda_2)^2}\right), \quad \text{(A1)}$$

$$E^{s,4}_0 = -N\left((2n-2)(2n-4)^2\frac{h_1^4}{(\lambda_1+\lambda_2)^3} + \ldots\right).$$

A well defined $1/n$ expansion exists if condition (4.1) is fulfilled. Then the ground state energy can be written as

$$\frac{E^s_0}{N} = \varepsilon^{s,0}_0 + \frac{1}{2n}\varepsilon^{s,1}_0 + \frac{1}{(2n)^2}\varepsilon^{s,2}_0 + \ldots. \quad \text{(A2)}$$

The zeroth-order term is just

$$\varepsilon^{s,0}_0 = -\tfrac{1}{2}\lambda_1.$$

The first-order term is the sum of a geometric series, which stands in the first column in the right-hand sides of (A1). These elements have the property that in the $\langle 0|VgV \ldots gV|0\rangle$ perturbational expression the V perturbation always acts on the same spin. The sum of these terms is

$$\varepsilon^{s,1}_0 = -2nh_1\frac{2nh_1/(\lambda_1+\lambda_2)}{1-2nh_1/(\lambda_1+\lambda_2)} + 2nh_2.$$

The second-order term in (A2) is more complicated.

Weak-coupling expansion

The weak-coupling form of the Hamiltonian (paper I, equations (2.5) and (2.6)) is used in this calculation. The unperturbed part of the Hamiltonian is

$$H_0 = H_h + \sum_i (n-1)\lambda_2/2n$$

while the perturbation is

$$V = H_\lambda - \sum_i (n-1)\lambda_2/2n.$$

The ground state of H_0 is given by

$$\psi_0^{w,0} = |1'1' \ldots 1'\rangle$$

with the energy $E_0^{w,0} = -N[2(n-2)h_1 - (n-1)\lambda_2/2n]$.

The lower lying excitations contain a flipped pair of spins, and they are of two kinds:

$$\psi_{1,2k}^{w,0} = |1'1' \ldots 1'(2k)'(2n+2-2k)'1' \ldots 1'\rangle, \qquad k = 1, 2, \ldots, n,$$

with energy $E_{1,2k}^{w,0} = E_0^{w,0} + 4nh_1 + 4(h_2 - h_1)$, and

$$\psi_{1,2k+1}^{w,0} = |1'1' \ldots 1'(2k+1)'(2n+1-2k)'1' \ldots 1'\rangle, \qquad k = 1, 2, \ldots, n-1,$$

with energy $E_{1,2k+1}^{w,0} = E_0^{w,0} + 4nh_1$. The two excitations are Nn- and $N(n-1)$-fold degenerate, respectively. The higher lying excitations of H_0 contain more flipped spins.

The first few terms of the perturbational series for the ground state energy are

$$E_0^{w,0} = -N[(2n-2)h_1 - (n-1)\lambda_2/2n], \qquad E_0^{w,1} = 0,$$

$$E_0^{w,2} = -N\left[n\left(\frac{\lambda_1}{2n}\right)^2 \frac{1}{4nh_1 + 4(h_2 - h_1)} + (n-1)\left(\frac{\lambda_2}{2n}\right)^2 \frac{1}{4nh_1}\right],$$

$$E_0^{w,3} = -N\left(n(n-1)\frac{(\lambda_1/2n)^2\lambda_2/2n}{[4nh_1 + 4(h_2 - h_1)]^2} + 2n(n-1)\frac{(\lambda_1/2n)^2\lambda_2/2n}{[4nh_1 + 4(h_2 - h_1)]4nh_1}\right.$$
$$\left. + (n-1)(n-2)\frac{(\lambda_2/2n)^3}{(4nh_1)^2}\right).$$

The $1/n$ expansion can be written in the region (4.1) as

$$\frac{E_0^w}{N} = \varepsilon_0^{w,0} + \frac{1}{2n}\varepsilon_0^{w,1} + \frac{1}{(2n)^2}\varepsilon_0^{w,2} + \ldots. \tag{A3}$$

The zeroth-order term is

$$\varepsilon_0^{w,0} = -2nh_1 + \lambda_2/2.$$

The first-order term is the sum of elements where the V perturbation acts on the same pair of spins:

$$\varepsilon_0^{w,1} = 4nh_1 - \lambda_2 - \frac{1}{2}\frac{(\lambda_2 + \lambda_1)^2/8nh_1}{1 - (\lambda_2 + \lambda_1)/8nh_1} - \frac{1}{2}\frac{(\lambda_2 - \lambda_1)^2/8nh_1}{1 - (\lambda_2 - \lambda_1)/8nh_1}.$$

The higher-order terms in (A3) contain more pairs of flipped spins.

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